A GEOMETRIC MODEL OF BRAUER GRAPH ALGEBRAS

TAKAHIDE ADACHI AND AARON CHAN

Abstract. In this note, we give a construction of pretilting complexes for a Brauer graph algebra from certain curves on a surface.

1. Ribbon graphs

First recall the definition of a ribbon graph. A *ribbon graph* is a datum $\mathbb{G} = (H, \sigma_0, \sigma_1)$, where H is a finite set, $\sigma_0 : H \to H$ is a permutation, and $\sigma_1 : H \to H$ is a fixed-point free involution.

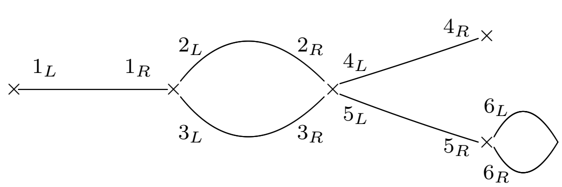
- The elements of H are called *half-edges* of $\mathbb{G}$.
- A $\langle\sigma_0\rangle$-orbit is called a *vertex* of $\mathbb{G}$. Let $s : H \to H/\langle\sigma_0\rangle$ be the canonical projection.
- A $\langle\sigma_1\rangle$-orbit is called an *edge* of $\mathbb{G}$. For each edge $E = \{h, \sigma_1(h)\}$, we call $s(h)$ and $s(\sigma_1(h))$ *endpoints* of E.
- The cycle of σ_0 corresponding to a vertex v is called the *cyclic ordering* around v.

A ribbon graph can be visualised like an ordinary graph by drawing edges incident to a vertex according to the cyclic ordering. We will always present the cyclic ordering in the clockwise direction. Let us present an example now.

Let $\mathbb{G} = (H, \sigma_0, \sigma_1)$ be the following ribbon graph.

- $H = \{i_L, i_R \mid i \in \{1, 2, \ldots, 6\}\}$.
- $\sigma_0 = (1_L)(1_R, 2_L, 3_L)(2_R, 4_L, 5_L, 3_R)(4_R)(5_R, 6_L, 6_R)$ as an element of the group of symmetries on H.
- σ_1 swaps i_L and i_R for any $i \in \{1, 2, \ldots, 6\}$.

Then we can draw $\mathbb{G}$ as follows.



We use this example to explain various notions throughout the rest of this note.

For a given ribbon graph $\mathbb{G} = (H, \sigma_0, \sigma_1)$, on one hand, we can associate to it a (multiplicity-free) Brauer graph algebra, and on the other hand, a decorated marked surface with arc system. We then use certain combinatorial objects on the surface to construct pretilting complexes for the Brauer graph algebra.

The detailed version of this paper will be submitted for publication elsewhere.

2. The Brauer graph algebra associated to a ribbon graph

Let us recall the definition of (multiplicity-free) Brauer graph algebras now.

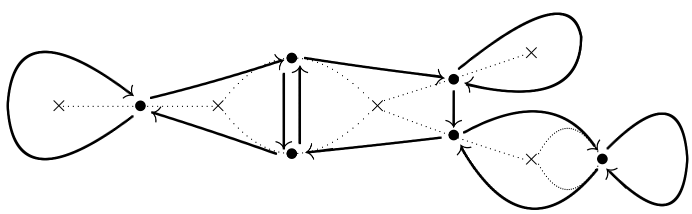
- There is a quiver $Q_{\mathbb{G}}$ associated to $\mathbb{G}$, where the set of vertices is given by $H/\langle\sigma_1\rangle$, and the set of arrows of $Q_{\mathbb{G}}$ is given by $\{(h|\sigma_0(h)) \mid h \in H\}$, that is, we draw an arrow $E \to E'$ for $E, E' \in H/\langle\sigma_1\rangle$ if there exist $h \in E$ and $h' \in E'$ such that $h' = \sigma_0(h)$.
- The *Brauer graph algebra associated to* $\mathbb{G}$ is the bounded quiver algebra $A_{\mathbb{G}} := KQ_{\mathbb{G}}/I_{\mathbb{G}}$, where K is a field and $I_{\mathbb{G}}$ is the two-sided ideal generated by relations of the following two types.

(Br1) $C_h - C_{\sigma_1(h)}$ for each $h \in H$, where C_e denotes the cycle of $Q_{\mathbb{G}}$ given by $(e|\sigma_0(e))(\sigma_0(e)|\sigma^2(e))\cdots(\sigma_0^{-1}(e)|e)$ (without repeating arrows).

(Br2) $(\sigma_0^{-1}(h)|h)(\sigma_1(h)|\sigma_0\sigma_1(h))$ for each $h \in H$.

Note that I_G is not necessarily an admissible ideal.

The following graph, where $\mathbb{G}$ is shown with dotted edges and crossed vertices, shows the quiver $Q_{\mathbb{G}}$ of our running example.



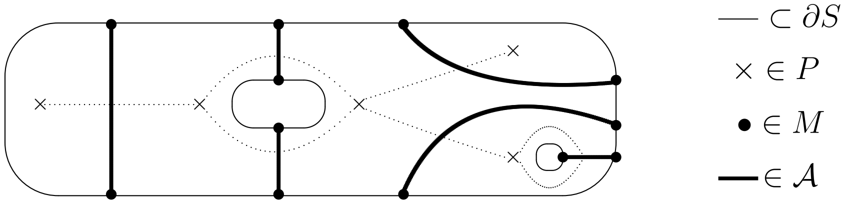
3. Decorated marked surface and arc system associated to a ribbon graph

Now we associate to a ribbon graph $\mathbb{G}$ a decorated marked surface (c.f. compare [1]) with arc system $\widetilde{S} = (S, P, M, \mathcal{A})$ as follows:

- S is a (compact, orientable) surface (with non-empty boundary ∂S) obtained by thickening $\mathbb{G}$, i.e., replace each edge of $\mathbb{G}$ by a square $[0,1] \times [0,1]$ and each vertex of $\mathbb{G}$ by a disc, and glue them according to the cyclic ordering. c.f. [2] for details.
- P is a set of distinguished points, called decorations, in the interior of S, identified with the embedding of the set $H/\langle\sigma_0\rangle$ of vertices of $\mathbb{G}$ in S.
- M is a set of distinguished points, called marked points, on the boundary of S given by the points $(1/2, 0), (1/2, 1)$ on every square that thickens an edge of $\mathbb{G}$. In particular, M is in bijection with H.
- $\mathcal{A}$ is a finite set of arcs dual to the set $H/\langle\sigma_1\rangle$ of edges $\mathbb{G}$ on S, i.e. for any $E \in \mathbb{G}$, there is a corresponding arc $\alpha_E \in \mathcal{A}$ given by the straight line $\{(1/2, t)\}_{t\in[0,1]}$ which connects the $(1/2, 0)$ and $(1/2, 1)$ in the square that thickens E. In particular, the endpoints of α_E are in M.

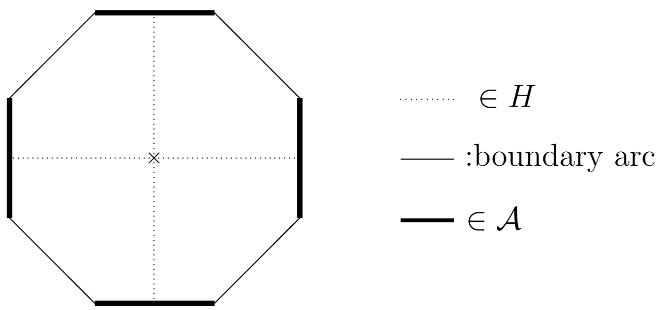
We concern only the topology ("combinatorics") of the decorated marked surface; the precise geometry of the decorated marked surface is not important to us. Moreover, for any curves or arcs on the surface, we always treat them up to isotopies (i.e., one can deform it as long as it does not crosses boundaries or decorations). Furthermore, if the endpoints of a curve is in P, then isotopies are rel endpoints; similar for curves with endpoints in M.

The following graph shows the decorated marked surface and arc system associated to our running example of $\mathbb{G}$.



4. From curves to complexes

Observe that cutting S along all the arcs in $\mathcal{A}$ gives a disjoint union of polygons, for which each of them looks like the following.



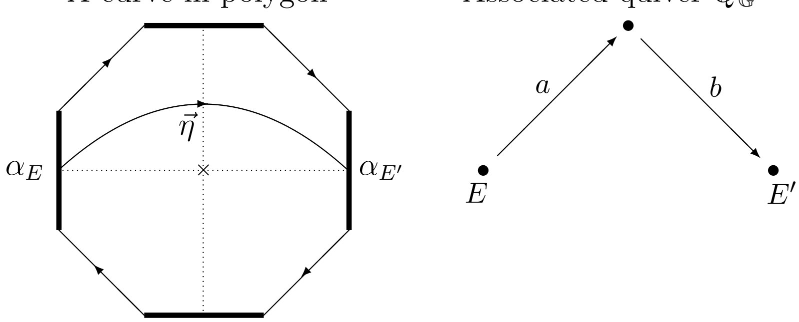
The orientation on the surface S induces orientation on the boundary arcs, which will be in the clockwise direction around each polygon. Observe that an oriented boundary arc starting in α_E and ending in $\alpha_{E'}$ corresponds to an arrow of $Q_{\mathbb{G}}$ from E to E'.

More generally, consider a curve $\vec{\eta}$ in a polygon that satisfies the following conditions.

- It is clockwise-oriented;
- It starts in $\alpha_E \in \mathcal{A}$ and ends in $\alpha_{E'} \in \mathcal{A}$;
- up to isotopies that keep endpoints of $\vec{\eta}$ in α_E and $\alpha_{E'}$, $\vec{\eta}$ does not intersect with itself.

Then one can identify $\vec{\eta}$ with a path that is non-vanishing in $A_{\mathbb{G}}$, hence with a map $P_E \to P_{E'}$; see the picture below.

A curve in polygon | Associated quiver $Q_{\mathbb{G}}$ | Maps in $A_{\mathbb{G}}$-proj



$$P_E \xrightarrow{-\cdot ab} P_{E'}$$

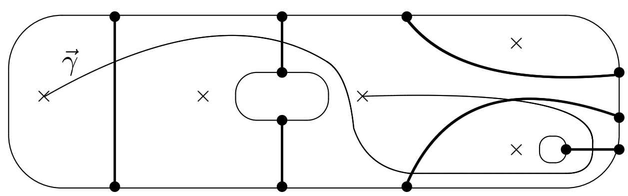
Consider now a curve $\vec{\gamma} : [0,1] \to S$ that intersects P only at its endpoints, intersects ∂S nowhere, and also self-non-intersecting except at the endpoints. Then cutting S along arcs of $\mathcal{A}$ effectively partition $\vec{\gamma}$ into segments of curves. Each segment is either a curve in the form we have described in the previous paragraph, or a curve with one endpoint in P and one endpoint in an arc of $\mathcal{A}$. Discarding the segments of the latter type, then by the above correspondence of clockwise-oriented curves in polygons and maps between projective modules, we get a sequence of maps of projective modules.

More precisely, suppose we have

$$\vec{\gamma} \cap \mathcal{A} = \{\vec{\gamma}(t_1), \vec{\gamma}(t_2), \dots, \vec{\gamma}(t_k)\}$$

with $0 < t_1 < t_2 < \cdots < t_k < 1$. Then for each $i \in \{1, 2, \dots, k\}$ we have an indecomposable projective $A_{\mathbb{G}}$-module $T_{\vec{\gamma}}^{(i)}$ corresponding to the arc that $\vec{\gamma}(t_i)$ lives in. Recall that $\mathcal{A}$ is in bijection with the set of vertices of $Q_{\mathbb{G}}$. The subcruve $\vec{\gamma}|_{[t_i,t_{i+1}]}$ of $\vec{\gamma}$ then corresponds to a map between $T_{\vec{\gamma}}^{(i)}$ and $T_{\vec{\gamma}}^{(i+1)}$. By the relation (Br2), this sequence of maps yields a complex of projective $A_{\mathbb{G}}$-modules. Note that if $|\vec{\gamma} \cap \mathcal{A}| = 1$, then we just get an indecomposable projective module which is regarded as a complex concentrated in a single degree.

Let us look at an example of this construction.



$\vec{\gamma}$ yields the complex:
$$\begin{tikzcd} P_5 \arrow[r, "(5_R|6_L)"] & P_6 \arrow[r, "(6_R|5_R)"] & P_5 \arrow[r, "(5_L|3_R)(3_R|2_R)"] & P_2 \\ & & P_1 \arrow[ur, "(1_R|2_L)"'] & \end{tikzcd}$$

Note that since the maps between projectives always given by multiplying a path, for typographical simplicity, we just denote them using only the path in the example here.

5. Main result

To state our main result, we need to be more precise about the degree that the complex lives in. We retain the notation and assumptions on $\vec{\gamma}$. As described, there is a projective $A_{\mathbb{G}}$-module

$$T_{\vec{\gamma}} := \bigoplus_{i=1}^{k} T_{\vec{\gamma}}^{(i)}.$$

For an integer n, we can naturally define a complex $T_{\vec{\gamma},n}$ such that its underlying module is $T_{\vec{\gamma}}$ and the degree of $T_{\vec{\gamma}}^{(1)}$ is n. Define an equivalence relation $\sim$ given by $(\vec{\gamma}, n) \sim (\vec{\gamma}', m)$ if $T_{\vec{\gamma},n} = T_{\vec{\gamma}',m}$. We call an equivalence class of a pair $(\vec{\gamma}, n)$ a *graded string.*

Suppose $(\vec{\gamma}_1, n_1), (\vec{\gamma}_2, n_2)$ are two (possibly the same) graded strings. We call them an *admissible pair* if the following conditions are satisfied.

- If $\vec{\gamma}_1(0) = \vec{\gamma}_2(0)$, then $n_1 = n_2$.
- If $\vec{\gamma}_1(0) = \vec{\gamma}_2(1)$, then $n_1 = n_2'$, where $(\vec{\gamma}_2, n_2) \sim (\vec{\gamma}_2', n_2')$.

- If $\vec{\gamma}_1(1) = \vec{\gamma}_2(0)$, then $n_1' = n_2$, where $(\vec{\gamma}_1, n_1) \sim (\vec{\gamma}_1', n_1')$.
- If $\vec{\gamma}_1(1) = \vec{\gamma}_2(1)$, then $n_1 = n_2$.
- Up to isotopy rel endpoints, we have $\vec{\gamma}_1 \cap \vec{\gamma}_2 \subset P$.

A graded string is admissible, if it forms an admissible pair with itself. More generally, a finite set $\Gamma := \{(\vec{\gamma}_1, n_1), (\vec{\gamma}_2, n_2), \ldots, (\vec{\gamma}_k, n_k)\}$ of graded strings is *admissible* if all elements are admissible, and pairwise admissible to each other.

Theorem 1. *If Γ is an admissible set of graded strings, then*

$$T_\Gamma := \bigoplus_{(\gamma_i, n_i) \in \Gamma} T_{\vec{\gamma}_i, n_i}$$

is a pretilting complex for $A_{\mathbb{G}}$, i.e. the Hom-space $\operatorname{Hom}(T_\Gamma, T_\Gamma[n])$ in the bounded homotopy category of finitely generated projective $A_{\mathbb{G}}$-modules vanishes for all non-zero n.

References

[1] Y. Qiu, *Decorated marked surfaces: spherical twists versus braid twists*, Math. Ann. **365** (2016), 595–633.

[2] R. J. Marsh, S. Schroll, *The geometry of Brauer graph algebras and cluster mutations*, J. Algebra **419** (2014), 141–166.

Takahide Adachi
Graduate school of Science
Osaka Prefecture University
1-1 Gakuen-cho, Nakaku, Sakai, Osaka, 599-8531, JAPAN
E-mail address: adachi@mi.s.osakafu-u.ac.jp

Aaron Chan
Graduate School of Mathematics
Nagoya University
Furocho, Chikusaku, Nagoya 464-8602, JAPAN
E-mail address: aaron.kychan@gmail.com